Another Variation on the Lee-Carter Model

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The Lee-Carter model for age-specific vital rates, introduced in 1992 (Lee and Carter 1992), has received much attention and has been extended and applied in many ways. Originally applied to age-specific mortality rates, the model provides a framework for representing in a compact parametric way a historical series of age-specific rates, and, with a few additional assumptions, conducting a stochastic forecast of vital rates through extrapolation of the fitted model. The model has also been applied to fertility-rate data and for making population projections (Lee and Tuljapurkar 1994; Booth and Tickle 2003). The model has also been shown to produce more accurate forecasts than a number of other approaches (Bell 1997).

It is important to distinguish between model *specification* and parameter *estimation* when discussing the Lee-Carter (henceforth, LC) model. As originally proposed, the specification entails two equations. The first equation decomposes a time series of age-specific vital rates into three sets of parameters: one set of parameters represents the age pattern of rates; a second set represents a time series of innovations or "shocks" that act on the schedule of rates; and the third a set of parameters that represent age-specific reactions to the period-specific shocks. The second equation is a model of the time path of the period-specific shocks. The original LC paper adopted a combination of singular value decomposition estimation for the first equation and time-series methods for modeling the evolution of the period-specific shocks. Since 1992 a number of variations on LC have been proposed. Some of the proposed variations pertain to model specification, for example relaxing the stationarity and linearity assumptions embedded in the original formulation (Booth, Maindonald, and Smith 2002; Carter and Prskawetz 2001); others propose alternative estimation approaches including maximum likelihood (Wilmoth 1993) or Poisson regression (Wilmoth 1993; Brouhns, Denuit, and Vermunt 2002).

This paper proposes yet another variation on LC. By combining the two equations that constitute the LC model into one, I derive a simplified model—one that nevertheless preserves all features essential to forecasting. I show that the LC model has the same form as random-factor models used in structural equation modeling; in particular, it is a one-factor model. It is straightforward to generalize the one-factor model to include additional random factors. And, if all possible cross-equation error covariances are included in unrestricted form, the model has the same form as a seemingly unrelated regression (SUR) model. These generalizations add parameters to the model, while leaving unchanged the predicted mean trajectory of forecasted rates. The variations proposed here allow for progressively more complex representations of the cross-equation correlations of the stochastic components of the forecasts. Thus, their primary effect is to change the range of forecast uncertainty.

The Basic Model

As originally proposed, and most often applied, the LC model contains two equations and is estimated in two stages. The first equation expresses an element of an age-time matrix of vital rates as a combination of a pure age effect, a_x, the interaction of a scalar period effect with an age effect, $b_x k_t$, and an age-time specific error term, as follows:

$$\ln[m_{xt}] = a_x + b_x k_t + \varepsilon_{xt}. \qquad (1)$$

The second equation represents the time path of period effects as a simple random walk:

$$k_t = k_{t-1} + c + e_t \qquad (2)$$

or, in difference form,

$$k_t - k_{t-1} = c + e_t. \qquad (2')$$

In (2) c represents average annual drift and e_t is pure noise, assumed to follow a normal distribution with mean zero and variance σ_1^2. The c parameter forms the basis for extrapolative forecasts of the mean path of vital rates, while the variance of e is used in the construction of error bounds on those forecasts.

The usual approach for estimating the LC model consists of first setting each a_x to the mean value of its respective time series, then subtracting a_x from each series and applying the singular value decomposition (SVD) to the residualized matrix in order to obtain estimates of the b_x and k_t vectors. Two identifying normalizations are required at this step; those used by LC are $\Sigma_x b_x = 1$ and $\Sigma_t k_t = 0$. LC also disregard the first-stage errors. Thus the first stage of estimation can be thought of as a data-reduction step, in which G (ages or age groups) × T (time periods) observed values are represented using $2G + T - 2$ parameters. Despite the large reduction in degrees of freedom, this decomposition works very well in practice, typically reproducing 95 or more percent of the variability in the original data. Estimation of the second equation entails the use of standard time-series methods, and it represents the T period effects using 2 parameters (c and the variance of e; however for purposes of forecasting an initial value for k must be specified as well).

A Variation on the Basic Model

This paper is concerned with the implications of recasting the LC model in first-differences form. To simplify notation, let $y_{xt} = \ln[m_{xt}]$ and $y^*_{xt} = y_{xt} - \varepsilon_{xt}$, i.e. the approximation to the true log-rate obtained using the SVD. Then, using (1) and (2) we obtain

$$y^*_{xt} - y^*_{x,\,t-1} = b_x c + b_x e_t, \qquad (3)$$

or, using Δ to represent the first-differences operator,

$$\Delta y^*_{xt} = b_x c + b_x e_t, \qquad (3')$$

The differencing approach eliminates the a_x parameters from the model. However, as pointed out in Lee (2000) it is not necessary to equate the a_x parameters to the respective means of each time series of rates. An alternative is to equate them to $\ln[m_{x,T}]$, the observed (log) rates in the final period, which in turn represent the natural "jumping-off" point for any forecast. But the final-period log-rates are observed, not estimated, and therefore need not be viewed as model parameters. Thus, (3) [or (3′)] preserves all the essential elements of the LC model for forecasting purposes.

We can stack up all G equations and write (3′) in vector form as

$$\Delta \mathbf{Y}^{*}_{t} = \mathbf{B}_1 + \boldsymbol{\omega}_t, \tag{4}$$

where $\mathbf{B}_1 = [\ \dots, b_x c, \dots\ ]'$ and $\boldsymbol{\omega}_t = [\ \dots, b_x e_t, \dots\ ]'$. Expression (4) describes a conventional linear system of equations, although the dependent variables are the first differences of approximations to the originally observed data. Moreover the LC model structure implies the restrictions

$$E(\boldsymbol{\omega}_t\, \boldsymbol{\omega}_t') = \sigma_1^2 \mathbf{B}_2 \mathbf{B}_2' \tag{5}$$

and

$$\mathbf{B}_1 c = \mathbf{B}_2; \tag{6}$$

that is, the covariances of residuals are strongly patterned (in particular, any pair of rows or of columns are a fixed multiple of each other) while the structural coefficients and the elements of the covariance decomposition are, as well, related by a fixed constant of proportionality. Given these restrictions, along with the assumed normality of e_t, (4) is of the form of a "measurement" or confirmatory factor analysis model, one with a single random factor. The $\mathbf{B}_2$ parameters are factor loadings, and all the randomness in the G equations arises from the scalar normally-distributed random factor e. Another important feature of the LC model is that the correlations between all pairs of period shocks ω_i and ω_j are identically one.

However, if it is possible to eliminate the a_x parameters from the LC model without compromising one's ability to perform forecasts, it is also unnecessary to eliminate the first-stage errors—the approximation error introduced through use of the SVD—when the model is recast in first-differences form. In particular, using y_{xt} instead of y^*_{xt} the first-differences model can be written as

$$\Delta y_{xt} = b_x c + b_x e_t + v_{xt},$$

where $v_{xt} = \varepsilon_{xt} - \varepsilon_{x,t-1}$, in other words as an error-components model. Having reintroduced the "approximation error" while maintaining the distinction between it and the "structural" error e_t, it is necessary to impose some assumption about the form of the former error. I assume that the ε_{xt} have constant variance and are uncorrelated over time and across equations, such that $E[\mathbf{v}_t \mathbf{v}_t']$ is diagonal. This is justified—admittedly somewhat casually—by the argument that if the approximation errors are small enough to be disregarded altogether, then an incorrect assumption that their covariances are zero is unlikely to do much damage.

The error components model for differences in log observed rates can be written in vector form as

$$\Delta \mathbf{Y}_t = \mathbf{B}_3 + \mathbf{B}_4 e_t + \mathbf{v}_t, \qquad (7)$$

an intercept-only model in which $\mathbf{B}_3$ is a vector of average annual changes in log observed rates and $\mathbf{z}_t = \mathbf{B}_4 e + \mathbf{v}_t$ is a vector of composite errors with covariance matrix

$$E[\mathbf{z}_t \mathbf{z}_t'] = \sigma_2^2 \mathbf{B}_4 \mathbf{B}_4' + \mathbf{\Phi}, \qquad (8)$$

with $\mathbf{\Phi}$ diagonal. Identification requires some normalization, e.g., $\sigma_2 = 1$. The structure of (8) preserves, at least conceptually, the distinction between the structural (dyanamic) errors and the approximation errors found in LC, despite the fact that the SVD need not be applied and, therefore, the approximation errors are implicit rather than explicit.

It is possible to impose the additional restriction that $\mathbf{B}_3$ and $\mathbf{B}_4$ differ by only a constant of proportionality, producing a slight generalization of LC, i.e., one that simply reintroduces the previously disregarded approximation errors. Note that there is no inherent need to normalize the $\mathbf{B}_3$ vector in (7) (in contrast to the b_xs in LC) because the observed data—the sample means of y_x—identify these intercepts. Moreover, if the LC constraint $\mathbf{B}_3 = \delta\mathbf{B}_4$ is imposed, it is no longer necessary to impose the normalization $\sigma_2 = 1$. However, with the scale factor σ_2 unconstrained the random-factor version of LC can be estimated, with no loss of generality, by imposing the constraint $\mathbf{B}_3 = \mathbf{B}_4$. The sum of the elements in the $\mathbf{B}_3$ vector should be approximately equal to the c parameter in the original LC setup [equation (2)] while the estimated variance of σ_2 should be approximately equal to $\Sigma B_3/\sigma_1$. Without the LC constraints imposed, (7) and (8) entail the estimation of $3G$ parameters; with them imposed, there are $2G + 1$ parameters to estimate.

The model in (7) and (8) serves as a natural basis for exploring more complicated models, namely through the introduction of additional random factors. With additional such factors the model continues to be a form of measurement or confirmatory factor-analytic model. With *m* orthogonal normally distributed random factors $u_1, \ldots, u_m$, for example, each normalized to have a variance of one, the expression for a single equation from the system is

$$\Delta y_{xt} = b_{3x} + \psi_{1x}u_1 + \ldots + \psi_{mx}u_m + v_{xt},$$

or, in vector form,

$$\Delta\mathbf{Y}_t = \mathbf{B}_3 + \mathbf{\Psi U} + \mathbf{V}_t \qquad (9)$$

(I continue to denote the intercepts using $\mathbf{B}_3$ because the intercepts in (9) will be identical to those in (7)]. In (9) the covariance of the residuals is $\mathbf{\Psi\Psi}' + \mathbf{\Phi}$. As progressively more factors are added to the model, it should do a progressively better job of reflecting the empirical covariances of regression residuals, and the variances of the "approximation error" components (represented by the diagonal matrix $\mathbf{\Phi}$) should become smaller. The orthogonality of the random factors is not restrictive, inasmuch as any multivariate normal distribution can be represented as a linear transformation of a vector of independent normal variables. One can easily imagine that several ex post identifiable factors could be identified using this approach; for example, if sex- and race-specific rates were jointly modeled, there might emerge race- and sex-specific factors, as well as factors for clusters of ages (e.g., young, middle-aged, and old) as well as an overall shared factor. Each additional random factor added to (9) produces G more parameters to estimate. There are as many as $G(G + 1)/2$ estimable parameters in the covariance matrix of regression residuals. Estimation of such models, using "covariance analysis" software (such as LISREL) requires that the data be arranged as T vector-valued period observations.

An altogether different approach to generalizing the LC model is to ignore the distinctions among error components, writing the equations for first differences in logs of each series as an intercept (i.e. the mean annual change) plus a scalar error, i.e.,

$$\mathbf{Y}_{xt} = \mathbf{D}\mathbf{B}_3 + \mathbf{U}_{xt}, \qquad (10)$$

for x = 1, … , G age groups and t=1, … T time periods. In contrast to the factor-analytic approaches of equations (7) and (9), in (10) there are $G \times T$ observations, i.e. the data for each age-group series are stacked. **D** is a $(G \times T)$-by-G matrix of dummy variables that equal one when the dependent variable pertains to the indicated age group, and zero otherwise. Error terms are assumed to be uncorrelated across observations within age groups, but to share a common covariance structure across age groups, that is

$$E(u_{x,\,t}\, u_{x+j,\,t}) = \sigma_{x,\,x+j}.$$

With these assumptions, (10) is a seemingly unrelated regressions (SUR) model (Greene 2000), which can be estimated using generalized least squares. Note that if the cross-equation correlations are assumed to be zero, producing a series of G independent age-specific regressions, we have the "naïve" model used as a baseline in Bell's (1997) study of the comparative performance of different forecasting approaches. However, in the SUR approach, the empirical cross-equation covariance structure is reproduced exactly. The price of this accuracy is a substantial increase in the number of parameters: the SUR model produces G estimated intercepts and $G(G + 1)/2$ estimated variance and covariance terms. Interestingly, it is common to decompose the cross-equations error correlation matrix $\mathbf{\Sigma}$ using the Cholesky root, $\mathbf{A}$, a $G \times G$ matrix with the property $\mathbf{A}'\mathbf{A} = \mathbf{\Sigma}$. Sampling from the distribution of period shocks is equivalent to sampling from $\mathbf{u}_t = \mathbf{A}'\mathbf{z}_t$ where $\mathbf{z}_t$ is a G-vector of standard normal random variables. This representation of the period-shock vector closely resembles the error-components expressions used in the factor-analytic version of the model, e.g. (9).

Thus, the first-differences approach can be used to specify a number of different models:

(i) differences in log approximate rates, with LC restrictions imposed [equation (4)];

(ii) differences in log approximate rates, adding additional random factors (a possible generalization that I did not discuss);

(iii) differences in log rates, with a single random factor plus diagonal approximation errors, and LC constraints imposed (i.e., LC with the approximation errors included) [equations (7) and (8)];

(iv) differences in log rates, with single random factor plus diagonal approximation errors but without LC restrictions imposed;

(v) differences in log rate, with two or more random factors plus diagonal approximation errors [equation (9)]; and

(vi) differences in log rates with unrestricted covariance matrix of residuals (SUR) [equation (10)].

Note that variants (iv) – (vi) all produce identical estimates of the average annual change in age-specific log rates—estimates that equal average observed year-to-year changes over the historical period of the data used—and thus differ only with respect to their treatment of the covariance matrix of residuals. Accordingly, the final three variants of the model will produce identical mean forecast trajectories; however, they can be expected to differ with respect to the width of error bounds on those forecasts. In this paper I present results for (iii) – (v), and forecasts of life expectancy based on (iv) – (vi).

Empirical Analysis

I present estimates based on U.S. death-rate data for male males and white females for the period 1933-2001. The data pertain to 19 age groups, 0-1, 1-4, 5-9, … , 80-84, and 85+ years old, producing a total of 38 data series. Thus there are 38 observed age-specific mean rates (averaging over time) and 38 × 39 / 2 = 741 observed variances and covariances. The SUR model therefore produces 38 + 741 = 779 parameters.

Table 1 presents the average annual changes in log rates for the 38 age-sex groups, along with their standard deviations and *p*-values for tests of the null hypothesis that the annual changes are normally distributed. Note that the means shown in Table 1 are identical to the estimated intercepts of models (7), (9), and (10). The annual rate of decline in log-rates can be seen to be greater for women than for men, in 17 of 19 age groups. Moreover, the annual reductions in mortality rates are greater, on average, in younger age groups and smallest among the elderly. The standard deviations of annual changes in log-rates are large relative to the mean, large enough to imply that positive rather than negative annual changes (i.e. rising rather than falling) death rates are not uncommon. Investigation reveals that a few of the largest SDs result from special circumstances; for example, the largest SD in Table 1 (for men age 20-24) can be explained by the presence of two outlier values, one a large increase from 1942 to 1943, and the other a large drop from 1945 to 1946. These are clearly associated with World War II combat mortality, for which this age group of men were at particular risk. I have not, however, introduced into the models any controls for exceptional circumstances such as these. Outliers such as those just described may, as well, be responsible for several of the cases in which the null hypothesis of normality is rejected. The null hypothesis fares better for the women's death rates (11 of 19 age groups) than for the men's (9 of 19).

Table 2 summarizes the correlations of deviations from means for the 38 age-sex specific mortality rate series. These correlations are, as well, the cross-equation error correlations for the SUR model, equation (10). To save space while capturing the essence of the patterns, I have recoded correlations into ranges; a minus sign indicates a negative correlation, a zero a correlation in the range 0.0 to 0.1, a 1 a correlation in the range 0.1 to 0.2, and so on. Also I have shaded correlations of 0.6 and above (and, completely darkened cells in which the correlations are 0.8 or more) in order to highlight clusters of highly-correlated year-to-year shocks in mortality rates. Correlations for the 19 age groups among men appear in the upper left triangle, cross-sex correlations in the upper right square, and across age group correlations among women are in the lower right triangle of Table 2. Unsurprisingly, nearly all correlations are positive, but the majority are modest (i.e. less than 0.5). The largest correlations are for adjacent or near-adjacent age groups, but this pattern is most pronounced for older ages, 50-54 and older. This pattern is repeated within and between the two sexes.

Table 3 presents selected parameter estimates from 3 different model specifications: LC with approximation errors [equation (7) with equality constraints imposed]; generalized LC with one random factor [equation (7) without equality constraints imposed]; and a two-factor version [equation (9)]. The SUR model is completely described by the column of means in Table 1 and the array of cross-equation correlations summarized in Table 2. For each model I also show the SDs of the so-called “approximation” errors, i.e. error variability not accounted for by the common random factor[s]. All three models were estimated using SAS’s PROC CALIS.

The factor loadings (which are also the negative of the intercepts) for the constrained LC model are all significantly different from zero. They are also, in most cases, troublingly different from the empirical means of the annual changes in log-rates (which possibly casts doubt on the correctness of the estimation approach taken here). A large reduction in χ^2 is achieved by relaxing the equality constraints ($\chi^2 = 104.99$ with 37 df, $p < 0.00001$), and another substantial reduction is achieved through adding a second random factor ($\chi^2 = 398.83$ with 37 df, $p < 0.00001$). A linear dependency emerged in the 2-factor model, necessitating the imposition of a zero constraint on one of the factor loadings. Nearly all factor loadings are significant in the unconstrained one-factor model. An interesting pattern emerges in the two-factor model, however. Here, the first factor clearly is associated with young through middle age, with the largest numerical values and the preponderance of statistically significant values appearing in those age groups. The second factor, in contrast, is primarily associated with the middle to oldest age groups. Because both factors pertain to both sexes, this model structure will induce both cross-age groups and cross-sex-age group correlations in the sample paths of year-to-year changes in log rates. Finally, there is a fairly uniform pattern of reduction in the “noise” part of the model, i.e. the error variation not accounted for by the common factors (the **v**s), as the complexity of the model is increased.

Forecasts of Life Expectancy

Because of the proven utility of the LC model for forecasting, I have used three variant forms of the first-differences model to forecast life expectancy at birth, for years 2002 – 2050. I used simulation techniques for the forecasts. In all cases the same mean path (the Bs) is used, and treated as fixed. Thus, the forecasts differ only with respect to the treatment of the stochastic errors. Projected mortality for age group x in year $T + h$ equals $m_{x, T} + \exp(hB_x + \Sigma_{t = 1, \ldots, h} v_{xt})$. I conducted forecasts using the 1-factor (without LC constraints) and 2-factor models presented in Table 3, as well as for the SUR model. Results for the 2-factor and SUR models are virtually indistinguishable. In order to minimize the effects of Monte Carlo variation, I used the same vector of randomly-generated standard normal variables to drive all three variant forms of each forecast, for both sexes. I conducted 1000 independent forecasts and averaged the results. Life expectancy is calculated using single-year-of-age mortality rates. I took single-year death rates for ages 0-99 for 2001 from Arias (2004). A linear regression of log rate on age, for ages 40-99, was used to extrapolate death rates to age 119.

Forecast results are summarized in Table 4 and Figure 1. The 1-factor model predicts growth in life expectancy at birth from 74.8 years (for men) in 2002 to 84.79 years in 2050, a growth of about 10 years in life expectancy, which is in turn a 13.4% increase in life expectancy at birth. This projection is well above that presented in Lee and Carter (1992), who projected growth in life expectancy at birth (for men and women combined) from 77.49 years in 2000 to 84.34 years in 2050, an increase of just 6.85 years (which represents an 8.84% increase from the 2000 figure).

The standard deviation of life expectancy, computed using the 1000 independent forecasts as data, show that the 2-factor model produces larger SDs than does the 1-factor model. The 2-factor model, in turn, imposes fewer restrictions on the covariance of period-specific shocks across age groups than does the 1-factor model; and, the 1-factor model is much less restrictive than LC which, as noted before, requires all such correlations to equal one. However, the excess of the SDs from the 2-factor model over those from the 1-factor model diminishes over time, and has effectively vanished by the end of the forecast period. Finally, Table 4 also shows the correlation coefficient between the random variables "forecasted male life expectancy" and "forecasted female life expectancy." The 1-factor and 2-factor models produce virtually the same results for these correlations, and furthermore the correlations, while quite large, diminish gradually over the forecast period. The 1-factor model produces more highly correlated forecasts of male and female life expectancy in the early years of the forecast, reflecting the smaller relative variability in the forecasts produced by that version of the model. Figure 1 illustrates the mean pathway of male life expectancy as well as 95% confidence bounds for life expectancy, computed by adding and subtracting 1.96 SDs, using both the 1-factor and 2-factor model results.

Discussion

This paper explores a variant form of the Lee-Carter model for mortality rates, based on a first-differences specification that combines Lee and Carter's two equations. In the first-differences version the a_x parameters that summarize the age pattern of mortality are eliminated. Those parameters are not, however, needed to produce forecasts; forecasting, in turn, has been demonstrated to be a major strength of LC. I show that the LC model in first-differences form has the same structure as a one-factor measurement or confirmatory factor-analysis model. However, in that form the LC model imposes a rather strong assumption on the age-specific deviations from their respective mean trajectories of year-to-year change, namely that they are all perfectly correlated. This strong assumption can be relaxed in two ways, first by reintroducing to the model the "approximation errors" that are disregarded in the usual LC model, and second by relaxing the assumption that correlations across age-specific deviations are driven by a single random factor. In the limit, there are as many random factors as there are ages (or age groups), leading to a seemingly-unrelated regressions form of the model. While it may be debatable whether the SUR model is still some form of the original Lee-Carter specification, LC is clearly a special case of it.

The results presented here must be viewed as a first step, for they leave many questions unanswered. For example, the versions of the model used to produce the forecasts presented here appear to predict considerably more growth in life expectancy at birth than does the LC model. A possible reason for this difference lies in the fact that LC model year-to-year changes in approximate rather than actual log rates. But, I have not tried to replicate the true LC specification here, and therefore say more about these differences. One avenue for further work is to compute the approximate values of log mortality rates using the SVD, as in Lee and Carter (1992), then use first differences of those approximate values as the dependent variables for a 1-factor model as in equation (4). Another interesting extension would be to introduce race differences. I have constructed a series of age- and sex-specific mortality rates for nonwhites for the same time period (1933-2001), and it would be interesting to investigate the factor structure of year-to-year changes in age-sex-race specific death rates. This, however, is a formidable task inasmuch as the data has fewer observations (on the time axis, i.e. 68 observations) than it has dependent variables ($2 \times 2 \times 19 = 76$), creating difficult identification problems.

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Table 1
Means, SDs, and Tests for Normality: Average Changes in log Mortality Rates

	Mean	SD	p for H0:Y normal
Males:			
0 - 1	-0.0340	0.0512	0.000
1 - 4	-0.0385	0.0522	0.415
5 - 9	-0.0354	0.0624	0.000
10 - 14	-0.0287	0.0581	0.000
15 - 19	-0.0127	0.0553	0.503
20 - 24	-0.0120	0.0800	0.000
25 - 29	-0.0146	0.0598	0.000
30 - 34	-0.0153	0.0491	0.000
35 - 39	-0.0134	0.0419	0.000
40 - 44	-0.0136	0.0316	0.006
45 - 49	-0.0130	0.0232	0.035
50 - 54	-0.0126	0.0224	0.119
55 - 59	-0.0122	0.0229	0.764
60 - 64	-0.0113	0.0212	0.120
65 - 69	-0.0098	0.0202	0.126
70 - 74	-0.0096	0.0229	0.005
75 - 79	-0.0091	0.0255	0.209
80 - 84	-0.0084	0.0266	0.128
85 plus	-0.0049	0.0436	0.122
Females:			
0 - 1	-0.0335	0.0508	0.000
1 - 4	-0.0399	0.0586	0.101
5 - 9	-0.0346	0.0635	0.000
10 - 14	-0.0299	0.0676	0.000
15 - 19	-0.0218	0.0683	0.001
20 - 24	-0.0258	0.0504	0.000
25 - 29	-0.0264	0.0533	0.002
30 - 34	-0.0240	0.0427	0.079
35 - 39	-0.0200	0.0361	0.028
40 - 44	-0.0183	0.0308	0.291
45 - 49	-0.0170	0.0256	0.397
50 - 54	-0.0157	0.0216	0.990
55 - 59	-0.0148	0.0217	0.947
60 - 64	-0.0139	0.0185	0.840
65 - 69	-0.0130	0.0201	0.150
70 - 74	-0.0135	0.0213	0.932
75 - 79	-0.0132	0.0271	0.029
80 - 84	-0.0117	0.0286	0.140
85 plus	-0.0061	0.0406	0.345

Table 2
Schematic Representation of Cross-Equation Error Correlations

Male (equations 1 - 19)																		Female (equations 20-38)																		
1[a]	3	1	1	-	-	0	-	0	1	1	-	0	0	-	1	-	-	[redacted]	2	1	3	-	2	1	1	1	-	0	-	0	0	-	-	1	-	-
	2	0	3	3	2	2	3	3	3	2	2	2	2	2	3	3	2	2	7	4	4	4	4	3	3	3	3	4	2	4	4	3	3	3	3	2
		3	4	2	3	3	3	1	2	2	1	2	1	0	1	-	-	3	2	3	2	4	1	5	4	2	1	1	3	2	2	1	0	1	-	0
			5	3	3	5	2	2	3	1	2	1	1	-	0	1	0	1	-	2	1	3	3	4	3	3	2	1	2	3	2	2	0	1	1	0
				6	5	5	4	4	4	3	3	3	3	2	3	3	1	1	4	3	4	4	5	4	5	3	4	2	3	4	3	3	1	3	2	1
					7	6	5	4	3	2	3	2	2	2	2	3	0	-	3	2	2	3	3	2	2	4	3	2	3	3	2	3	2	2	2	0
						7	7	5	4	2	3	1	1	1	2	2	0	-	2	1	1	3	4	3	3	4	3	3	3	3	2	2	0	2	1	0
							7	6	4	2	3	2	1	0	1	1	0	0	2	1	2	4	4	4	4	4	3	2	3	3	2	2	0	1	0	0
								7	5	4	4	3	2	1	3	2	2	-	3	1	2	4	3	4	5	5	4	5	5	3	3	3	1	2	1	2
									7	6	5	5	4	3	4	3	3	0	3	1	4	4	5	4	5	6	6	5	5	4	4	5	4	4	3	3
										7	7	6	5	4	5	4	3	1	3	1	3	4	3	4	4	5	5	6	5	5	5	5	4	4	4	3
											7	7	6	6	6	5	4	0	2	1	2	3	2	3	2	3	4	5	6	4	5	5	5	4	5	4
												[redacted]	[redacted]	6	7	6	4	-	0	1	2	4	2	1	2	3	3	4	5	6	5	6	5	5	5	5
													7	7	7	6	4	0	1	1	3	4	1	2	2	2	2	4	5	5	6	5	6	5	5	4
														7	7	6	5	-	1	1	1	3	1	1	1	2	3	4	4	5	5	6	5	5	5	5
															6	6	6	-	1	1	2	3	1	0	1	1	3	4	4	5	5	6	7	4	6	6
																7	5	0	3	2	4	3	3	1	2	3	3	4	4	6	6	6	5	[redacted]	6	5
																	5	-	2	1	2	3	2	1	2	2	3	4	4	6	7	6	6	6	[redacted]	6
																		-	1	2	3	2	2	1	1	2	3	3	2	4	4	5	6	4	5	[redacted]
																			2	1	2	-	2	1	1	1	-	-	-	0	0	-	-	0	-	-
																				4	4	3	4	3	3	3	4	3	2	3	3	2	2	2	2	1
																					2	3	2	3	4	3	3	3	1	3	2	2	2	2	1	2
																						4	4	2	4	3	3	2	2	4	3	4	3	4	2	3
																							3	5	6	4	4	4	3	5	4	5	4	3	2	2
																								4	5	5	5	5	4	5	4	4	2	3	2	1
																									6	5	4	5	5	4	5	3	2	2	1	1
																										6	5	5	5	5	4	4	3	2	2	2
																											4	6	5	5	4	4	3	4	2	2
																												5	4	4	4	4	4	3	3	3
																													6	6	5	5	4	3	3	3
																														6	7	4	4	4	4	3
																															7	6	6	6	6	4
																																6	7	6	7	5
																																	[redacted]	6	7	6
																																		5	7	7
																																			7	4
																																				6

[a] Correlation of errors in equations 1 and 2

Note: - indicates correlation < 0; 0 indicates correlation between 0 and 0.1; and so on.

Table 3

Parameter Estimates for Three Variant Forms of Lee-Carter First-Differences Model

	1-Factor w/ LC cond.		1-Factor w/o LC cond.		2-Factor		
	$\mathbf{B}_3$ (= $\mathbf{B}_4$)[a]	SD(v)	$\mathbf{B}_4$	SD(v)	$\mathbf{\Psi}_1$	$\mathbf{\Psi}_2$	SD(v)
Males:							
0 - 1	0.0156	0.0562	0.0030	0.0511	0.0078	-0.0008	0.0506
1 - 4	0.0277	0.0471	0.0249 *	0.0458	0.0189 *	0.0188 *	0.0448
5 - 9	0.0232	0.0612	0.0167 *	0.0601	0.0306 *	0.0048	0.0541
10 - 14	0.0197	0.0566	0.0159 *	0.0559	0.0287 *	0.0053	0.0503
15 - 19	0.0194	0.0473	0.0285 *	0.0474	0.0338 *	0.0172 *	0.0402
20 - 24	0.0221	0.0725	0.0350 *	0.0719	0.0464 *	0.0199	0.0620
25 - 29	0.0178	0.0545	0.0237 *	0.0550	0.0424 *	0.0093	0.0412
30 - 34	0.0160	0.0446	0.0189 *	0.0454	0.0393 *	0.0051	0.0290
35 - 39	0.0163	0.0352	0.0218 *	0.0358	0.0315 *	0.0112 *	0.0252
40 - 44	0.0162	0.0225	0.0220 *	0.0228	0.0205 *	0.0157 *	0.0183
45 - 49	0.0135	0.0152	0.0172 *	0.0156	0.0115 *	0.0139 *	0.0146
50 - 54	0.0130	0.0148	0.0170 *	0.0146	0.0059 *	0.0159 *	0.0146
55 - 59	0.0132	0.0147	0.0182 *	0.0138	0.0041	0.0181 *	0.0134
60 - 64	0.0120	0.0143	0.0163 *	0.0136	0.0020	0.0168 *	0.0128
65 - 69	0.0107	0.0142	0.0151 *	0.0134	0.0004	0.0162 *	0.0119
70 - 74	0.0112	0.0170	0.0164 *	0.0159	-0.0028	0.0191 *	0.0122
75 - 79	0.0131	0.0170	0.0205 *	0.0152	0.0023	0.0211 *	0.0141
80 - 84	0.0129	0.0187	0.0209 *	0.0165	0.0229 *	[b]	0.0136
85 plus	0.0135	0.0382	0.0256 *	0.0354	-0.0037	0.0297 *	0.0318
Females:							
0 - 1	0.0144	0.0563	0.0008	0.0508	0.0082	-0.0032	0.0500
1 - 4	0.0278	0.0550	0.0235 *	0.0537	0.0232 *	0.0156 *	0.0515
5 - 9	0.0243	0.0608	0.0207 *	0.0600	0.0180 *	0.0145	0.0592
10 - 14	0.0278	0.0588	0.0321 *	0.0595	0.0220 *	0.0251 *	0.0588
15 - 19	0.0284	0.0552	0.0393 *	0.0559	0.0352 *	0.0278 *	0.0516
20 - 24	0.0234	0.0423	0.0254 *	0.0436	0.0299 *	0.0151 *	0.0378
25 - 29	0.0237	0.0456	0.0249 *	0.0471	0.0336 *	0.0127	0.0392
30 - 34	0.0213	0.0349	0.0228 *	0.0362	0.0278 *	0.0131 *	0.0297
35 - 39	0.0188	0.0280	0.0213 *	0.0291	0.0227 *	0.0137 *	0.0244
40 - 44	0.0162	0.0245	0.0182 *	0.0249	0.0143 *	0.0139 *	0.0235
45 - 49	0.0153	0.0184	0.0173 *	0.0189	0.0116 *	0.0138 *	0.0182
50 - 54	0.0137	0.0151	0.0152 *	0.0154	0.0098 *	0.0123 *	0.0148
55 - 59	0.0143	0.0125	0.0175 *	0.0128	0.0082 *	0.0153 *	0.0129
60 - 64	0.0128	0.0106	0.0152 *	0.0106	0.0045 *	0.0145 *	0.0106
65 - 69	0.0129	0.0120	0.0164 *	0.0117	0.0042 *	0.0160 *	0.0114
70 - 74	0.0128	0.0144	0.0162 *	0.0138	-0.0002	0.0178 *	0.0117
75 - 79	0.0146	0.0189	0.0199 *	0.0183	0.0034	0.0201 *	0.0179
80 - 84	0.0145	0.0203	0.0215 *	0.0189	-0.0015	0.0241 *	0.0153
85 plus	0.0145	0.0338	0.0266 *	0.0308	-0.0032	0.0304 *	0.0268

[a] All parameters significantly different from zero; [b] Parameter constrained to zero.

* $|t| > 1.96$

Table 4
Mean and SD of Projected e_0; Men and Women; 1-Factor and 2-Factor Models

	Males				Females					
	1-Factor		2-Factor		1-Factor		2-Factor		$\rho(e_0m, e_0f)$	
Year	e_0	$SD(e_0)$	e_0	$SD(e_0)$	e_0	$SD(e_0)$	e_0	$SD(e_0)$	1-Factor	2-Factor
2002	74.80	0.24	74.78	0.43	80.02	0.23	80.00	0.37	0.88	0.84
2003	75.05	0.49	75.05	0.60	80.30	0.44	80.30	0.53	0.85	0.83
2004	75.33	0.64	75.33	0.72	80.60	0.57	80.61	0.64	0.84	0.84
2005	75.59	0.77	75.58	0.85	80.89	0.68	80.89	0.74	0.84	0.83
2006	75.84	0.88	75.83	0.94	81.17	0.76	81.17	0.81	0.83	0.83
2007	76.09	0.96	76.09	1.03	81.44	0.83	81.44	0.89	0.83	0.82
2008	76.35	1.06	76.34	1.11	81.71	0.93	81.71	0.97	0.82	0.82
2009	76.59	1.14	76.56	1.20	81.98	1.00	81.96	1.05	0.82	0.82
2010	76.81	1.22	76.80	1.27	82.22	1.06	82.21	1.10	0.81	0.81
2011	77.05	1.29	77.05	1.34	82.48	1.12	82.48	1.15	0.81	0.81
2012	77.29	1.36	77.28	1.41	82.74	1.19	82.74	1.23	0.81	0.81
2013	77.51	1.43	77.52	1.46	82.99	1.25	83.00	1.26	0.81	0.80
2014	77.75	1.47	77.76	1.52	83.25	1.29	83.26	1.32	0.80	0.79
2015	78.00	1.52	77.99	1.55	83.50	1.33	83.50	1.36	0.79	0.78
2016	78.23	1.56	78.21	1.59	83.74	1.37	83.73	1.39	0.78	0.77
2017	78.44	1.60	78.43	1.65	83.97	1.41	83.97	1.45	0.77	0.77
2018	78.66	1.67	78.64	1.71	84.20	1.48	84.19	1.50	0.77	0.77
2019	78.88	1.72	78.87	1.75	84.43	1.52	84.43	1.54	0.76	0.76
2020	79.09	1.76	79.07	1.80	84.66	1.57	84.65	1.61	0.76	0.76
2021	79.29	1.80	79.28	1.84	84.88	1.61	84.87	1.65	0.75	0.75
2022	79.49	1.83	79.48	1.86	85.08	1.66	85.07	1.68	0.74	0.75
2023	79.69	1.87	79.68	1.90	85.28	1.69	85.28	1.71	0.74	0.74
2024	79.90	1.90	79.89	1.92	85.50	1.74	85.50	1.75	0.74	0.74
2025	80.11	1.93	80.10	1.96	85.71	1.77	85.71	1.80	0.73	0.73
2026	80.31	1.96	80.30	1.99	85.92	1.81	85.92	1.83	0.72	0.72
2027	80.50	2.02	80.50	2.05	86.13	1.86	86.15	1.88	0.72	0.71
2028	80.70	2.06	80.69	2.09	86.35	1.90	86.35	1.92	0.71	0.71
2029	80.90	2.10	80.91	2.13	86.56	1.93	86.56	1.95	0.70	0.69
2030	81.09	2.13	81.09	2.16	86.75	1.96	86.75	1.99	0.69	0.69
2031	81.29	2.17	81.29	2.21	86.95	2.00	86.95	2.03	0.69	0.68
2032	81.50	2.22	81.50	2.24	87.15	2.04	87.15	2.05	0.68	0.68
2033	81.70	2.25	81.69	2.26	87.35	2.08	87.34	2.10	0.67	0.67
2034	81.89	2.28	81.89	2.30	87.54	2.11	87.54	2.13	0.66	0.66
2035	82.07	2.30	82.07	2.32	87.73	2.15	87.73	2.18	0.65	0.65
2036	82.26	2.33	82.24	2.36	87.92	2.20	87.92	2.22	0.64	0.64
2037	82.43	2.38	82.42	2.41	88.10	2.25	88.09	2.27	0.65	0.65
2038	82.62	2.42	82.62	2.45	88.28	2.29	88.28	2.31	0.65	0.64
2039	82.81	2.44	82.81	2.45	88.47	2.32	88.47	2.33	0.64	0.63
2040	83.01	2.45	83.00	2.47	88.66	2.34	88.65	2.36	0.63	0.63
2041	83.19	2.49	83.20	2.50	88.83	2.39	88.84	2.43	0.63	0.63
2042	83.39	2.52	83.38	2.56	89.03	2.45	89.03	2.47	0.63	0.63
2043	83.55	2.57	83.54	2.59	89.20	2.50	89.20	2.53	0.63	0.63
2044	83.73	2.61	83.74	2.64	89.37	2.55	89.37	2.56	0.62	0.62
2045	83.92	2.65	83.92	2.67	89.54	2.58	89.54	2.60	0.62	0.62
2046	84.10	2.68	84.09	2.70	89.70	2.60	89.69	2.61	0.61	0.61
2047	84.26	2.72	84.26	2.77	89.85	2.63	89.86	2.67	0.61	0.61
2048	84.44	2.78	84.45	2.80	90.02	2.67	90.02	2.68	0.61	0.61
2049	84.63	2.81	84.62	2.84	90.18	2.69	90.17	2.72	0.60	0.60
2050	84.79	2.86	84.80	2.90	90.35	2.76	90.36	2.79	0.60	0.60

Figure 1: Mean and 95% Confidence Interval for Projected Male Life Expectancy