Decomposition of compositional effects: Explicating covariance in demographic analysis.(first draft)

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Abstract

Demographic dynamics can generally be decomposed into direct effects and compositional effects (due to changes in structures of the population). The magnitude of a compositional effect can often be measured by a rather obscure covariance term. We develop alternative formulas that permit decomposition of compositional effects into demographically meaningful components. We apply the new formulas to decompose change in life expectancy, crude birth and death rates, and the average age of a population.

The change over time in a demographic measure usually can be broken down into a direct effect due to change in the measure of interest and a compositional effect due to change in population structure or to heterogeneity in population structure. Two formulas used by demographers to evaluate compositional effects capture these effects by covariance terms. The meaning of these covariance terms can be difficult to intuitively understand. Furthermore, decomposition of the covariance can shed light on the nature of the compositional effect.

Hence in this article we develop several ways of breaking compositional effects down into demographically-meaningful components.

To start, it is useful to review some definitions and some notation. The weighted average of $v(x,y)$ over x will here be denoted by $\bar{v}(y)$, with

$$\begin{aligned}\bar{v}(y) &= \frac{\int_0^\infty v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx},\ x\ continuous, && (1.a)\\ &= \frac{\sum_x v_x(y)w_x(y)}{\sum_x w_x(y)},\ x\ discrete, && (1.b)\end{aligned}$$

where $v(x,y)$ is some demographic function of interest and $w(x,y)$ is some weighting function. The variable x can be continuous or discrete; the variable y is continuous. In the applications presented in this article, x sometimes denotes age and sometimes subpopulations whereas y is always time, but other application are also of interest.

The covariance function denoted $Cov_w(u,v)$ can be defined in terms of averages as

$$Cov_w(u,v) = \frac{\int_0^\infty \left[v(x,y)-\bar{v}(y)\right]\left[u(x,y)-\bar{u}(y)\right]w(x,y)dx}{\int_0^\infty w(x,y)dx}$$

$$= \frac{\int_0^\infty u(x,y)v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx} - \frac{\int_0^\infty u(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}\frac{\int_0^\infty v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}$$

$$= \overline{uv} - \bar{u}\bar{v}. \qquad (2)$$

As indicated by this equation, the covariance between two variables can be interpreted as measuring how much the mean of their product exceeds the product of their means. Note that for simplicity we often omit the arguments x, y and w.

Covariance as a measure of compositional change

Consider first the decomposition formula presented by Vaupel and Canudas Romo (2002). The equation can be simply and memorably expressed as

$$\dot{\bar{v}} = \bar{\dot{v}} + Cov(v, \acute{w}). \tag{3}$$

The change in the average, $\dot{\bar{v}}$, is

$$\dot{\bar{v}} = \frac{\partial}{\partial y} \frac{\int_0^\infty v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}. \tag{4}$$

The average change, $\bar{\dot{v}}$, is

$$\bar{\dot{v}} = \frac{\int_0^\infty \left[\frac{\partial}{\partial y} v(x,y)\right] w(x,y)dx}{\int_0^\infty w(x,y)dx}. \tag{5}$$

And the covariance $Cov(v, \acute{w})$ can be calculated as shown in (2).

The first term on the right-hand side of (3), the average change, might be called the direct component of change. The second component, the covariance between the variable of interest and the intensity of the weighting function, is the structural or compositional component of change. Vaupel and Canudas Romo (2002) note "in equation (3) the covariance is a measure of the extent to which the underlying variable of interest rises and falls with the relative derivative of the weighting function."

Other studies that find decompositions of the type of equation (3) where the covariance captures the compositional effect are the works by Preston, Himes and Eggers (1989) and Schoen and Kim (1991 and 1992).

To show how this decomposition of the change over time can be applied let the function $v(x,y)$ be equivalent to the force of mortality $\mu(a,t)$ at age a and time t and let the weighting function be $N(a,t)$ the age-specific population size over age a and time t. The crude death rate (CDR) $\bar{\mu}(t)$ can be calculated as

$$\bar{\mu}(t) = \frac{\int_0^{\omega} \mu(a,t) N(a,t) da}{\int_0^{\omega} N(a,t) da}. \tag{6}$$

It then follows directly from equation (3) that the change over time in CDR is decomposed as

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + Cov(\mu, r), \tag{7}$$

where $r(a,t)$ is the age-specific growth rate of the population which equals the intensity of the weighting function $r \equiv r(a,t) \equiv \acute{N}(a,t)$.

As an illustration of (7) is the decomposition of the change in the crude death rate for Germany from 1991 to 1997. Germany benefited from sizeable reductions in the CDR in the years after reunification, reducing its CDR from $\bar{\mu}(1991) = 11.397$ to $\bar{\mu}(1997) = 10.495$, per thousand. That is an annual change of $\dot{\bar{\mu}} = -0.150$. The German development is mainly due to the direct effect of large reductions in mortality, $\bar{\dot{\mu}} = -0.273$, particularly in the eastern part of Germany. The compositional effects pulled the CDR up by half of the direct effect,

$Cov(\mu, r) = 0.124$.

In many situations the demographic average in equation (1a) can be described as the product of two terms

$$\bar{v}(t) = \int_0^\omega v(x,t)c(x,t)dx, \tag{8}$$

where $c(x,t)$ denotes the proportion of the total values of the weights that belong to the category x at time t, $c(x,t) = \frac{w(x,t)}{\int_0^\infty w(x,t)dx}$, and therefore $\int_0^\omega c(x,t)dx = 1$. Under these circumstances equation (3) changes to

$$\dot{\bar{v}} = \bar{\dot{v}} + \overline{v\acute{c}}. \tag{9}$$

This is easily proved by looking at the derivative of $\bar{v}(t)$, which follows the rule of the derivative of a product

$$\dot{\bar{v}} = \int_0^\omega \dot{v}(x,t)c(x,t)dx + \int_0^\omega v(x,t)\dot{c}(x,t)dx = \bar{\dot{v}} + \overline{v\left(\frac{\dot{c}}{c}\right)}. \tag{10}$$

As discussed by Canudas Romo (2003) as a consequence of equations (3) and (10) the covariance component $C(v, \acute{w})$ for the compositional effect of change can also be expressed as

$$C(v, \acute{w}) = \overline{v\acute{c}}. \tag{11}$$

Covariance as a measure of compositional heterogeneity

The Second equation to study compositional effects is found by exchanging the product of averages, $\bar{u}\bar{v}$, in equation (2) from right to left obtaining

$$\overline{uv} = \bar{u}\bar{v} + Cov(u, v). \tag{12}$$

Vaupel and Canudas Romo (2003) utilize (12) to decompose the change over time of life expectancy, as discussed below. The formula, however, has many other uses. The basic idea is that the average of the product of two demographic variables can be decomposed into a direct effect and a compositional effect. The direct effect is the product of the average values of the two variables. The compositional effect captures population heterogeneity such that the two variables of interest tend to correlate across the segments of the population.

To see how this decomposition can shed light on change in life expectancy, let $\rho(a, t)$ denote the rate of progress in reducing death rates, $\rho(a, t) = -\acute{\mu}(a, t)$, where the force of mortality at age a and at time t is denoted by $\mu(a, t)$, and the remaining life expectancy at age a and time t is denoted by $e^o(a, t)$.

Vaupel and Canudas Romo show that the time-derivative of life expectancy at birth, denoted $\dot{e}^o(0, t)$, can be expressed as the average of the product $\rho(a, t)e^o(a, t)$. Applying (12) it is shown that

$$\dot{e}^o(0, t) = \bar{\rho}(t)e^{\dagger}(t) + Cov_f(\rho, e^o), \tag{13}$$

where $\bar{\rho}(t)$ and $e^{\dagger}(t)$ are the averages of $\rho(a, t)$ and $e^o(a, t)$ respectively with both averages weighted by the probability density function describing the distribution of deaths, $f(a,t)$, (i.e., lifespans) in the lifetable population at age a and time t.

The decomposition in equation (13) expresses the change in life expectancy at birth as the sum of two terms. The first term is the product of the average rate of mortality improvement, $\bar{\rho}(t)$, and the average number of life-years lost, $e^{\dagger}(t)$. This term captures the general effect of a reduction in death rates. The second term, the covariance between rates of mortality improvement and remaining life expectancies, increases or decreases the general effect, depending on whether the covariance is positive or negative. The covariance captures the effect of heterogeneity in $\rho(a,t)$ at different ages.

The covariance term is often hard to intuitively express. For instance, Vaupel and Canudas Romo (2003) comment "whether the covariance is positive or negative is determined by equation (2). It is difficult to capture the equation in a simple sentence. The basic idea is that the covariance will be positive if the age-specific pace of mortality improvement tends to be higher (or lower) than average at those ages when remaining life expectancy tends to be higher (or lower) than average— with the heaviest weights being given to those ages when death is most common. Remaining life expectancy generally declines with age. At some age a^*, $e^o(a^*,t) = e^{\dagger}(t)$. The covariance will be positive if before age a^* the age-specific pace of mortality improvement tends to be higher than average and if after age a^* the age-specific pace of mortality improvement tends to be lower than average."

As an illustration of (13) is the annual change in life expectancy at birth for the Swedish population in 1903, 1953 and 1998. Over the course of the 20^{th} century Swedish life expectancy increased substantially. The average pace of mortality improvement, $\bar{\rho}$, fluctuated from about 1.9% at the turn of the century to 2.1% at mid century and 1.6% at the end of the century.

The average number of life-years lost as a result of death, $e^{\dagger}$, dropped from 22 years in 1903 to around 12 years in 1953 and 10 years in 1998. The product $\bar{\rho}e^{\dagger}$ describes the increase in life expectancy due to the general advance in survivorship. This component is positive and is the main contributor to the increase in life expectancy. The compositional component is the covariance between age-specific improvements in mortality and remaining life expectancies. This term is positive but relatively small. More can be seen in Table 4 in this article.

In the next sections we introduce some alternative equations that can replace the covariance in equation (12) and (3).

Decomposing the covariance term

Four different decompositions of the compositional effect of change are shown. All these alternative equations are further refinements of the definition of the covariance in (2). Each subsection contains an application of the new decomposition.

Difference in growth rates: Change in the general fertility rate

The covariance, $Cov(v, \acute{w})$, is separated into two terms. The first term is the average of interest, $\bar{v}(y)$, and the second is the difference of two averages, $\left[\tilde{\acute{w}}(y) - \bar{\acute{w}}(y)\right]$. Each average of the intensity in the weighting function $\acute{w}(x, y)$ is weighted by different terms. $\bar{\acute{w}}(y)$ is, as before, weighted by the function $w(x, y)$ while $\tilde{\acute{w}}(y)$ includes as weight the product $v(x, y)w(x, y)$. The covariance is decomposed as $Cov(v, \acute{w})$

$$Cov(v, \acute{w})$$

$$
\begin{aligned}
&= \frac{\int_0^\infty v(x,y)\acute{w}(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx} - \frac{\int_0^\infty v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}\frac{\int_0^\infty \acute{w}(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx} \\
&= \bar{v}\left[\frac{\int_0^\infty v(x,y)\acute{w}(x,y)w(x,y)dx}{\int_0^\infty v(x,y)w(x,y)dx} - \frac{\int_0^\infty \acute{w}(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}\right] \\
&= \bar{v}\left[\tilde{\acute{w}} - \bar{\acute{w}}\right].
\end{aligned} \tag{14}
$$

As shown in the case of the crude death rate in equation (7), in many cases the weighting function, $w(x,y)$, equals $N(a,t)$, the age-specific population size over age a and time t. As a consequence the intensity of the weighting function equals the age-specific growth rate of the population $r(a,t) \equiv \acute{N}(a,t)$. The difference $\left[\tilde{\acute{w}}(y) - \bar{\acute{w}}(y)\right]$ can then be seen as a difference of two overall population growth rates, $\left[\tilde{\acute{w}}(y) - \bar{\acute{w}}(y)\right] = [\tilde{r}(y) - \bar{r}(y)]$. This difference in growth rates represents how much (or less) is the product of the variable of interest and the population size, $V(a,t) = v(a,t)N(a,t)$, growing respect to the average growth of the population.

Let $v(x,y) = b(a,t)$ denote the age-specific birth rate, let $w(x,y) = N_f(a,t)$ be the age-specific female population size and let $\bar{g}(t)$ be the general fertility rate (GFR), which is simply the number of babies divided by the number of women at reproductive ages. The change in this rate is given by (3) and (14) as

$$
\dot{\bar{g}} = \dot{\bar{b}} + \bar{g}\left[\tilde{r}_f - \bar{r}_f\right], \tag{15}
$$

where $\tilde{r}_f(y)$ is the average growth in births if they had experienced the same growth as the rest of the population. Table 1 shows calculations based on equation (15) that decompose the change in the GFR for China, Denmark and Mexico from 1990 to 1995. Table 1 indicates that the GFR fell in China and Mexico and rose in Denmark. In all three countries the

Table 1: General fertility rate, $\bar{g}(t)$, in percentage, and the decomposition of the annual change over time from 1990 to 1995, for China, Denmark and Mexico.

	China	Denmark	Mexico
$\bar{g}(1990)$	7.871	4.850	11.083
$\bar{g}(1995)$	6.283	5.373	9.671
$\dot{\bar{g}}(1992.5)$	-0.317	0.105	-0.282
$\bar{\dot{b}}$	-0.280	0.081	-0.286
$Cov(b, r_f)$	-0.036	0.023	0.004
$\bar{g}$	7.036	5.078	10.345
$\tilde{r}$	0.802	0.285	2.640
$\bar{r}$	1.317	-0.165	2.600
$\bar{g}\,[\tilde{r} - \bar{r}]$	-0.036	0.023	0.004
$\dot{\bar{g}} = \bar{\dot{b}} + \bar{g}\,[\tilde{r}_f - \bar{r}_f]$	-0.316	0.104	-0.282

Source: Authors' calculations described in the Note, based on U.S. Census Bureau (2001).

dominant component of this shift was the average change in age-specific birth rates. Changes in age-composition, captured by the covariance term, had a relatively minor impact, especially in Mexico. It could be naively thought that there was no change in the structure of the population. Hence, to better explicate this component we have included here the alternative decomposition in (15).

The population growth rate, $\bar{r}(t)$, is the highest in Mexico followed by China while the Danish females in reproductive ages experienced a decline in the studied period. For Denmark the average growth weighted by the number of babies is positive as a result of greater number of babies in those age groups that experienced some increase. Resultant of this is the positive covariance of 0.023 in Denmark. The opposite occurs in China with lower alternative average, $\tilde{r}(t)$, than for the whole population of women, $\bar{r}(t)$. This difference of growth rates adjusts the level of the GFR. The observed covariances between age-specific birth rates and growth rates is mainly a consequence of the distinct average growth rates $[\tilde{r} - \bar{r}]$.

Decomposition (14) of the compositional effect of change explicates that is through adjusting the average $\bar{v}(t)$ with a difference of average growth that the structure of the population influences the total change in $\bar{v}(t)$.

Difference in averages: Change in the average age of the population

The covariance function $Cov(u, v)$ in (2) is a commutative function, that is both variables u and v are similarly and can exchange places for $Cov(v, u)$. Equation (14) can also be reexpressed in terms of the average of the intensity of the weighting function, $\bar{\acute{w}}(y)$, and the difference of the averages of the variable of interest weighted by different terms, $[\tilde{v}(y) - \bar{v}(y)]$. The covariance is decomposed as

$$Cov(v, \acute{w}) = \bar{\acute{w}} \left[\tilde{v} - \bar{v}\right]. \tag{16}$$

Preston, Himes and Eggers (1989) showed that the change over time of the average age of the population, $\bar{a}(t)$, can be expressed as $\dot{\bar{a}} = Cov(a, r)$. Alternatively we could write using (16)

$$\dot{\bar{a}} = \bar{r} \left[\tilde{a} - \bar{a}\right], \tag{17}$$

where $\bar{r}(t)$ is the growth rate of the population, $\bar{a}(t)$ is the average age of the population, and $\tilde{a}(t)$ corresponds to the average age of the population that experienced the change, $\dot{N}(a, t) = r(a, t)N(a, t)$.

Table 2 illustrates equation (16), by determining the decomposition of the change in the average age of the population, using the same countries and period as Preston, Himes and

Eggers.

Table 2: Average age of the population, $\bar{a}(t)$, and the decomposition of the annual change over time from 1970 to 1980 for Japan, the Netherlands and the United States.

	Japan	Netherlands	United States
$\bar{a}(1970)$	31.586	32.634	32.378
$\bar{a}(1980)$	33.969	34.632	34.035
$\dot{\bar{a}}(1975)$	0.238	0.200	0.166
$Cov(a, r)$	0.239	0.200	0.166
$\bar{r}$	0.012	0.009	0.011
$\tilde{a}$	51.986	57.296	48.527
$\bar{a}$	32.765	33.677	33.240
$\bar{r}\,[\tilde{a} - \bar{a}]$	0.239	0.200	0.166
$\dot{\bar{a}} = \bar{r}\,[\tilde{a} - \bar{a}]$	0.239	0.200	0.166

Source: Authors' calculations described in the Note, based on U.S. Census Bureau (2001). The data for the Netherlands corresponds for the years 1971 to 1981.

Japan leads in population growth with a $\bar{r}(t)$ of 1.2%, followed by the United States with 1.1% and the Netherlands 0.9%. In the Netherlands the elderly experienced the greatest changes and as a consequence the alternative average age $\tilde{a}(t)$ is of 57.3 years. The contrary is seen in the U.S. where the increase in younger groups contributes to lower the gap between the average ages $\tilde{a}(t)$ and $\bar{a}(t)$.

The new decomposition of the compositional effect of change confirms the suspicion of the highest growth rates among the aged population. Furthermore, the change can also be explained as an adjustment of the average growth rate by the difference between the average of the population experiencing change and the total population.

Product of differences: Change in the world's life expectancy

Another alternative covariance decomposition proposed here below includes three terms. The first component is the average of the positive deviations of $v(x, y)$ from its mean, $[v - \bar{v}]_+$. The second term corresponds to the difference of population growth for the age-groups with positive and negative deviations of $v(x, y)$ from its mean, this term is expressed as $\left[\tilde{\acute{w}}_+ - \tilde{\acute{w}}_-\right]$. Finally is the proportion of age-groups with positive deviations of $v(x, y)$ from its mean, denoted π_+.

Following the definition in equation (2) the covariance can also be expressed as

$$Cov(v, \acute{w}) = \frac{\int_0^\infty \acute{w}(x, y) \left[v(x, y) - \bar{v}(y)\right] w(x, y) dx}{\int_0^\infty w(x, y) dx}. \tag{18}$$

Two indicator functions are used to separate positive from negative values of the difference between the function of interest and its average, $v(x, y) - \bar{v}(y)$. These indicator functions are denoted $I_+(v - \bar{v})$ and $I_-(v - \bar{v})$ for the positive and negative cases respectively. The function $I_+(v - \bar{v})$ has values of one when the values of $v(x, y) - \bar{v}(y)$ are positive and zero otherwise and analogous for the negative values of $I_-(v - \bar{v})$. From (18) we have

$$Cov(v, \acute{w}) = \frac{\int_0^\infty \acute{w}(x, y) \left[v(x, y) - \bar{v}(y)\right] \left[I_+ \left[v - \bar{v}\right] + I_- \left[v - \bar{v}\right]\right] w(x, y) dx}{\int_0^\infty w(x, y) dx}. \tag{19}$$

Defining the positive average of the deviations of $v(x, y)$ from its mean as

$$[v - \bar{v}]_+ = \frac{\int_0^\infty \left[v(x, y) - \bar{v}(y)\right] I_+ \left[v - \bar{v}\right] w(x, y) dx}{\int_0^\infty I_+ \left[v - \bar{v}\right] w(x, y) dx},$$

and the proportion of these positive averages as

$$\pi_+ = \frac{\int_0^\infty I_+ [v - \bar{v}] \, w(x,y) dx}{\int_0^\infty w(x,y) dx},$$

we can rewrite the covariance from (19) as

$$Cov(v, \acute{w}) =$$

$$\frac{\int_0^\infty \acute{w}(x,y) \left[v(x,y) - \bar{v}(y)\right] \left[I_+ [v - \bar{v}] + I_- [v - \bar{v}]\right] w(x,y) dx}{\int_0^\infty \left[v(x,y) - \bar{v}(y)\right] I_+ [v - \bar{v}] \, w(x,y) dx} \left[v - \bar{v}\right]_+ \pi_+. \qquad (20)$$

Let the alternative average of the weighting function be expressed as

$$\tilde{\acute{w}}_+ = \frac{\int_0^\infty \acute{w}(x,y) \left[v(x,y) - \bar{v}(y)\right] I_+ [v - \bar{v}] \, w(x,y) dx}{\int_0^\infty \left[v(x,y) - \bar{v}(y)\right] I_+ [v - \bar{v}] \, w(x,y) dx},$$

analogous for the negative terms $\tilde{\acute{w}}_-(t)$. Then the first term in (20) can be further separated into

$$\frac{\int_0^\infty \acute{w}(x,y) \left[v(x,y) - \bar{v}(y)\right] \left[I_+ [v - \bar{v}] + I_- [v - \bar{v}]\right] w(x,y) dx}{\int_0^\infty \left[v(x,y) - \bar{v}(y)\right] I_+ [v - \bar{v}] \, w(x,y) dx} = \tilde{\acute{w}}_+ - \tilde{\acute{w}}_-, \qquad (21)$$

this is done by adding and subtracting the terms $\tilde{\acute{w}}_- - \tilde{\acute{w}}_-$. As a result of (20) and (21) the covariance is decomposed as

$$Cov(v, \acute{w}) = \left[\tilde{\acute{w}}_+ - \tilde{\acute{w}}_-\right] \left[v - \bar{v}\right]_+ \pi_+. \qquad (22)$$

Age heterogeneity is only one of the multitudinous dimensions of population heterogeneity - some observed and some unobserved. In this section we present an example of averages over another characteristic, namely nationality.

Consider a population composed of different subpopulations. The life expectancy at birth at time t for the entire population, $\bar{e}^o(t)$, is the average over the subpopulations' life expectancy at birth

$$\bar{e}^o(t) = \frac{\sum_i e_i^o(t) N_i(t)}{\sum_i N_i(t)}, \tag{23}$$

where $N_i(t)$ is the size of subpopulation i and $e_i^o(t)$ is the subpopulation life expectancy at birth. The change in $\bar{e}^o$ over time can be decomposed utilizing (3) and (22) as

$$\dot{\bar{e}}^o = \bar{\dot{e}}^o + [\tilde{r}_+ - \tilde{r}_-] \left[e^o - \bar{e}^o\right]_+ \pi_+, \tag{24}$$

where $r_i(t)$ is the population growth rate of the ith subpopulation, $r_i(t) \equiv \acute{N}_i(t)$.

In Table 3 formula (24) is applied to changes in life expectancy of the world population. The world experienced an increase in life expectancy with an annual change of more than three months per year ($\dot{\bar{e}}^o(1985) = 0.26$). The covariance between life expectancy and population growth rates among the subpopulations is modest. Because the covariance is negative, the countries with long life expectancy tend to have slow rates of population growth. On average 6.3 years are observed from the population experiencing higher longevity than average. Opposed to this are the 8 years from those countries below average $[e^o - \bar{e}^o]_-$. Both groups, above and below average $\bar{e}^o$ have proportions around 50 %, as seen in the π_+. The compositional

Table 3: Life expectancy at birth, $\bar{e}_o(t)$, for the world and decomposition of the annual change over time in life expectancy from 1980 to 1990.

	World
$\bar{e}^o(1980)$	62.790
$\bar{e}^o(1990)$	65.401
$\dot{\bar{e}}^o(1985)$	0.261
$\bar{\dot{e}}^o$	0.314
$Cov(e^o, r)$	-0.053
$[\tilde{r}_+ - \tilde{r}_-]$	-0.015
$[e^o - \bar{e}^o]_+$	6.301
π_+	0.560
$[\tilde{r}_+ - \tilde{r}_-]\,[e^o - \bar{e}^o]_+\,\pi_+$	-0.053
$\dot{\bar{e}}^o = \bar{\dot{e}}^o + [\tilde{r}_+ - \tilde{r}_-]\,[e^o - \bar{e}^o]_+\,\pi_+$	0.261

Source: Authors' calculations described in the Note, based on World Bank data (2001). The subpopulations are the populations of the countries of the world for which data were available.

component is negative due to the difference $[\tilde{r}_+ - \tilde{r}_-] = -0.015$, resultant of growth rates in the countries below average life expectancy. The increase in life expectancy of the world is thus lower than the average increase in national life expectancy.

Decomposition (22) of the compositional effect of change goes further into detail of the observed change. As shown in the application above any of the three main terms can explain the level and the direction of the change. Therefore, it is crucial to go on this level of detail of the decomposition to clearly disentangle the reasons of this dynamic in the demographic variables.

Product of differences: Change in life expectancy

Utilizing the commutative property of the covariance, as in equation (16), we can exchange the places of v and $\acute{w}$ in equation (22). The alternative compositional effect is written as

$$Cov(v, \acute{w}) = [\tilde{v}_+ - \tilde{v}_-] \left[\acute{w} - \bar{\acute{w}}\right]_+ \pi_+. \tag{25}$$

As an application of these equation we can replace the compositional effect of change in life expectancy in (13) for

$$\dot{e}^o(0,t) = \bar{\rho}(t)e^\dagger(t) + \left[\tilde{\rho}_+ - \tilde{\rho}_-\right] \left[e^o - e^\dagger(t)\right]_+ \pi_+. \tag{26}$$

Table 4 shows the application of equation (26) to the annual change in life expectancy at birth for the Swedish population in 1903, 1953 and 1998.

The remaining life expectancy reduces with age and the positive values of $\left[e^o - e^\dagger(t)\right]_+$ are found in the younger age-groups. This average number of remaining life expectancy above average declines over time. The improvement in mortality in the older age-groups together with the increment in remaining life expectancy below average, $\left[e^o - e^\dagger(t)\right]_-$, act in the difference $\left[\tilde{\rho}_+ - \tilde{\rho}_-\right]$. The increase observed in the fifties in the $\left[\tilde{\rho}_+ - \tilde{\rho}_-\right]$ slows down in the last decade of the twentieth century. The compositional component passes from representing less than 10% of the total change in life expectancy to above 15% along with the twenty century.

We conclude with this application because it is a clear example of an increasing importance of the compositional effect overtime. The alternative decomposition (25) is a clear explanation to the obscure covariance used in (13).

Table 4: Life Expectancy at birth, $e^o(0,t)$, and the decomposition of the annual change around the first of January of 1903, 1953 and 1998, in Sweden.

t	1903	1953	1998
$e^o(0,t-2.5)$	52.239	71.130	78.784
$e^o(0,t+2.5)$	54.527	72.586	79.740
$\dot{e}^o(0,t)$	0.458	0.291	0.191
$\bar{\rho}$ (%)	1.852	2.083	1.587
$e^\dagger$	22.362	11.988	10.053
$\bar{\rho}e^\dagger$	0.414	0.249	0.159
$Cov_f(\rho, e^o)$	0.044	0.042	0.032
$\left[\tilde{\rho}_+ - \tilde{\rho}_-\right]$	0.005	0.010	0.010
$\left[e^o - e^\dagger(t)\right]_+$	23.048	14.870	10.175
π_+ (%)	36.900	27.800	30.800
$\left[\tilde{\rho}_+ - \tilde{\rho}_-\right]\left[e^o - e^\dagger(t)\right]_+ \pi_+$	0.044	0.042	0.032
$\dot{e}^o(0) = \bar{\rho}e^\dagger + \left[\tilde{\rho}_+ - \tilde{\rho}_-\right]\left[e^o - e^\dagger(t)\right]_+ \pi_+$	0.458	0.291	0.191

Source: Authors' calculations described in the Note. Life table data is derived from the Human Mortality Database (2002). Life table values for the years 1900 and 1905, 1950 and 1955, 1995 and 2000, were used to obtain results for the mid-points around January 1, 1903, 1953 and 1998.

Discussion

In many situations in population studies the interest centers in the relationship or association between demographic variables. One measure of linear dependency is the covariance.

The covariance function is found in many studies in demography. To cite a few studies are Mauskopf and Wallace (1984) analyzing the covariance between children born and child deaths across women and the study of Casterline et al. (1986) looking at the covariance between the ages of wife and husband in several countries. Also among these group of research are David and Sanderson (1987) taking into account the covariance between couple's fecundability and their overall health status and David et al. (1988) looking at the covariance between age at marriage and extent of fertility control.

Here we focused in another group of studies that have also employed the covariance between demographic variables. We refer to those studies of the change over time of demographic variables. Among the research on the dynamics of population are the work by Preston, Himes and Eggers (1989) expressing the change over time in the average age of the population as the covariance between ages and age-specific growth rates. Extensions of the result by Preston, Himes and Eggers also led to the study of change over time of other variables by Schoen and Kim (1992) and a general decomposition method presented by Vaupel and Canudas Romo (2002). Both studies include covariances to express the change in demographic variables due to change in the structure of the population. Furthermore, the decomposition of the change over time in life expectancy presented by Vaupel and Canudas Romo (2003) also includes a covariance between the improvement of mortality and the remaining life expectancy. Finally is the work by Schoen and Kim (1991) where the change over time of the distance to stability is expressed also as a covariance.

Whether it is a covariance or not, this is the change due to compositional effect of change. Here, we presented several alternative expressions for explicating the covariance term of the compositional effect of change. The alternative decompositions are presented in equations (14), (16), (22) and (25).

Decompositions (14) and (16) add four terms to the change: the demographic average $\bar{v}(t)$ and the population growth rate $\bar{r}(t)$ at the mid points, and the two alternative averages for the variable of interest $\tilde{v}(t)$ and the population growth $\tilde{r}(t)$. The first two have straight forward interpretation and they are used as reference levels that are altered by the product of a difference of two terms. In these differences is where the alternative averages are included. Both $\tilde{v}(t)$ and $\tilde{r}(t)$ can be interpreted as the demographic average and growth in the period of change.

Decompositions (22) and (25) include five new terms, grouped into three components. These are two alternative averages $\tilde{v}(t)$ and $\tilde{r}(t)$, with positive and negative cases. Two averages of deviations from their means, $[v(a,t) - \bar{v}(t)]_+$ and $[r(a,t) - \bar{r}(t)]_+$, and finally the proportion of positive cases $\pi_+(t)$. The commutative property of the covariance permits the election of the functions $v(x,y)$ and $\acute{w}(x,y)$ in the alternative decompositions of $Cov(v, \acute{w})$. Therefore it is possible to utilize either equation (22) or (25) to explicate the compositional effect. Since the weighting function $w(x,y)$ is in many cases the population size $N(a,t)$ then the intensity $\acute{w}(x,y)$ is equal to the population growth $r(a,t) = \acute{N}(a,t)$. Equation (22) is preferred over (25) because the age-specific growth rates fluctuate along the ages, as shown by Canudas Romo (2003), as a result the difference $[r(a,t) - \bar{r}(t)]$ those not have an age pattern. Similar situation is seen for the age-specific improvement in mortality $[\rho(a,t) - \bar{\rho}(t)]$ and then the decomposition shown in (26) is preferred.

Another definition of the covariance that includes variances is

$$Cov(u,v) = \frac{1}{2}\left[var(u) + var(v) - var(u-v)\right]. \tag{27}$$

Equation (27) represents one half of the extent in which the variance of the difference exceeds the addition of the variances. Equation (27) could also lead to further research on explicating the use of covariance in demography.

Note

If data are available for time t and $t{+}h$, then we generally used the following approximations for the value at the mid-point $t + h/2$. For the relative derivative of the function $v(a, t + h/2)$, we used

$$\acute{v}(a, t + h/2) \approx \frac{\ln\left[\frac{v(a,t+h)}{v(a,t)}\right]}{h}. \tag{28}$$

The value of the function at the mid-point $v\left(a, t + h/2\right)$ was estimated by

$$v(a, t + h/2) \approx v(a, t)e^{(h/2)\acute{v}(a,t+h/2)}. \tag{29}$$

Substituting the right-hand side of (28) for $\acute{v}(a, t + h/2)$ in (29) yields the equivalent approximation

$$v(a, t + h/2) \approx \left[v(a, t)v(a, t + h)\right]^{1/2}. \tag{30}$$

This is a standard approximation in demography (Preston, Heuveline and Guillot, 2001). The derivative of the function $v(a, t + h/2)$ was estimated by

$$\dot{v}(a, t + h/2) = \acute{v}(a, t + h/2)v(a, t + h/2). \tag{31}$$

We used (28), (29) and (31) wherever we thought that the rate of change was more or less constant over the time interval. In some cases it seemed appropriate to assume that change in the interval was linear. This was the case when we estimated the change over time in the survival function $\ell(a,t)$ and in life expectancy $e^o(a,t)$ in Table 4. Then we used

$$v(a,t+h/2) \approx \frac{v(a,t+h)+v(a,t)}{2} \tag{32}$$

and

$$\dot{v}(a,t+h/2) \approx \frac{v(a,t+h)-v(a,t)}{h}. \tag{33}$$

The period force of mortality in an interval was calculated using an equation similar to (28). If data are available for ages a and $a+k$ we used the following approximation

$$\mu(a+k/2,t) \approx -\frac{\ln\left[\frac{\ell(a+k,t)}{\ell(a,t)}\right]}{k}. \tag{34}$$

The rate of progress in reducing death rates $\rho(a+k/2,t+h/2)$ was calculated as

$$\rho(a+k/2,t+h/2) = -\acute{\mu}(a+k/2,t+h/2) \approx -\frac{\ln\left[\frac{\mu(a+k/2,t+h)}{\mu(a+k/2,t)}\right]}{h}. \tag{35}$$

Because the force of mortality in (34) and $\rho(a+k/2,t+h/2)$ in (35) are at ages $a+k/2$, it was necessary to calculate the other functions involved in the decomposition at those ages. The survivorship function $\ell(a,t)$ and the remaining life expectancy $e^o(a,t)$ at age $a+k/2$ were calculated using an equation analogous to equation (30). The lifetable distribution of deaths was calculated as

$$f(a + k/2, t) \approx \mu(a + k/2, t)\ell(a + k/2, t). \tag{36}$$

For the estimation of some equations we substituted sums for integrals.

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